INVERSE BACKSCATTERING FOR THE SCHRÖDINGER EQUATION IN 2D

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ABSTRACT. We study the inverse backscattering problem for the Schrödinger equation in two dimensions. We prove that, for a non-smooth potential in 2D the main singularities up to 1/2 of the derivative of the potential are contained in the Born approximation (Diffraction Tomography approximation) constructed from the backscattering data. We measure singularities in the scale of Hilbertian Sobolev spaces.

1. INTRODUCTION

We consider the inverse scattering problem for the Schrödinger operator $H = -\Delta + q(x)$, with a real-valued potential $q(x)$. The scattering solution $u = u(k, \theta, x)$ associated with the energy k^2 and the incident direction θ is defined as the solution of the problem

$$\begin{cases} (-\Delta + q - k^2)u = 0\,, \\ u = e^{ikx\cdot\theta} + u_s\,, \end{cases} \tag{1}$$

where the function u_s satisfies the outgoing Sommerfeld radiation condition, which means, for a compactly supported potential q, that u has asymptotics as $|x| \to +\infty$

$$u(k, \theta, x) = e^{ikx\cdot\theta} + C|x|^{\frac{1-n}{2}} k^{\frac{n-3}{2}} e^{ik|x|} A(k, \theta, \theta') + o(|x|^{\frac{1-n}{2}})\,, \tag{2}$$

where $\theta' = \frac{x}{|x|}$. The function $A(k, \theta, \theta')$, $x \in \mathbb{R}$, θ, θ' in the unit sphere S^{n-1}, is called the scattering amplitude or far-field pattern.

The outgoing resolvent operator for the Laplacian is given, in terms of the Fourier transform, by

$$\widehat{R_k(f)}(\xi) = (-|\xi|^2 + k^2 + i0)^{-1}\widehat{f}(\xi)\,.$$

We obtain the so-called Lippmann-Schwinger integral equation by applying the outgoing resolvent to (1)

$$u_s(k, \theta, x) = R_k(q(\cdot)e^{ik(\cdot)\cdot\theta})(x) + R_k(q(\cdot)u_s(k, \theta, \cdot))(x)\,. \tag{3}$$

The key operator in the above integral equation is

$$T_k(f)(x) = R_k(q(\cdot)f(\cdot))(x)\,.$$

There are several *a priori* estimates for R_k that allow us to prove existence and uniqueness of Lippmann-Schwinger integral equation. Usually, Fredholm theory applies and everything follows

from compactness arguments, the Rellich uniqueness theorem and unique continuation principles, in the case of real-valued potentials. The solution can be obtained in several situations (these cases do not require q to be real) by perturbation arguments, assuming that the energy is sufficiently large, $k > k_0 \geq 0$, where k_0 depends on some *a priori* bound of the potential q. As an example, we may consider compactly supported $q \in L^r(\mathbb{R}^n)$ for some $r > \frac{n}{2}$. In this case, the resolvent operator R_k is bounded from $L^p(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ with the norm decaying to 0 as $k \to \infty$ when $\frac{1}{p} - \frac{1}{p'} = \frac{1}{r}$, see [A], [KRS] and see also [R1]. This together with Hölder inequality proves that for big k the operator T_k is a contraction in L^p and then existence and uniqueness of solution of (3) easily follow.

Once the scattering solution is obtained we may prove that the far-field pattern can be expressed as

$$A(k, \theta, \theta') = \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) u(k, \theta, y) dy, \tag{4}$$

see [ER2] where this is used as a definition for the non-compactly supported q.

The Born series of q is obtained by inserting the Lippmann-Schwinger integral equation in (4)

$$A(k, \theta, \theta') = \hat{q}(k(\theta' - \theta)) + \sum_{j=2}^{\infty} \widehat{Q_j^\theta(q)}(k(\theta' - \theta)), \tag{5}$$

where

$$\widehat{Q_j^\theta(q)}(k(\theta' - \theta)) = \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} (qR_k)^{j-1}(q(\cdot)e^{ik\theta \cdot (\cdot)})(y) dy.$$

We deal with the backscattering inverse problem, for which one assumes the data with the direction of the receiver opposed to the incident direction (echoes), i.e $A(k, \theta, -\theta)$. The inverse problem is then formally well determined. The unique determination of q by the backscattering data is an open problem. Local and generic uniqueness have been proved by Eskin and Ralston [ER2], see also [S].

In this case, we obtain the Born series for backscattering data

$$A(k, \theta, -\theta) = \hat{q}(\xi) + \sum_{j=2}^{\infty} \widehat{Q_j(q)}(\xi), \tag{6}$$

where $\xi = -2k\theta$ and the j-adic term in the Born series is given by

$$\widehat{Q_j(q)}(\xi) = \int_{\mathbb{R}^n} e^{ik\theta \cdot y} (qR_k)^{j-1}(q(\cdot)e^{ik\theta \cdot (\cdot)})(y) dy.$$

We define the Born approximation for backscattering data as

$$\widehat{q_B}(\xi) = A(k, \theta, -\theta)$$

where $\xi = -2k\theta$.

The approximated potential q_B is the target of Diffraction Tomography. In this paper, we study how much information on the actual potential q can be obtained from q_B. We are able to prove that the main singularities of q and q_B are the same up to $1/2^-$ derivative.

A procedure to obtain this recovery of singularities from Diffraction Tomography is to give regularity estimates for the j-adic term in the Born series (6). The first of these estimates in 2D was obtained by Ola, Päivärinta and Serov, see [OPS], concerning the quadratic term Q_2 and was improved by Ruiz and Vargas, see [RV], who obtained the mentioned $1/2^-$ gaining of derivative for the

quadratic term. Nevertheless for q in the Sobolev space $W^{s,2}$ with $s > 1/2$, the known estimates for the term Q_3 in the Born series are not sufficient to assure that $q - q_B \in W^{\alpha,2}$ for $\alpha < s + 1/2$. To achieve this we prove the main result of this work.

Theorem 1. *Assume that q is a compactly supported function in $W^{s,2}(\mathbb{R}^2)$, for $0 \le s < 1$. Then $Q_3(q) \in W^{\alpha,2}(\mathbb{R}^2) + C^\infty(\mathbb{R}^2)$, for any $0 \le \alpha < s + 1$.*

This theorem, together with the Ruiz and Vargas estimate for the quadratic term and their estimates for the general j-adic term with $j > 3$, allows us to claim

Theorem 2. *Assume that q is a compactly supported function in $W^{s,2}$, for $0 \le s < 1$. Then $q - q_B \in W^{\alpha,2} + C^\infty$, for any $\alpha \in \mathbb{R}$ such that $0 \le \alpha < s + \frac{1}{2}$.*

We expect each term in the Born series (6) to win half a derivative with respect to the previous one, claiming the conjecture $Q_j(q) \in W^{\alpha_j,2}(\mathbb{R}^2) + C^\infty(\mathbb{R}^2)$, for all $\alpha_j \in \mathbb{R}$ with $0 \le \alpha_j < s + \frac{j-1}{2}$ $(j \ge 2)$, provided that q is a compactly supported function and $q \in W^{s,2}(\mathbb{R}^2)$, $0 \le s < 1$. We address this question in a future work.

The results in this paper and in [RV] could be extended for non-compactly supported potential assuming an appropriate decay at infinity. To simplify the matter we reduce ourselves to the compactly supported case.

Section 2 is the main one in this paper. We prove theorem 2 in Section 3. In Section 4, we include some lemmata often used in Section 2. In particular, lemma 4.4 is essential in order to get estimate (12).

Constants. We use the letter C to denote any constant that can be explicitly computed in terms of known quantities. The exact value denoted by C may therefore change from line to line in a given computation.

Notation. We will use the following notation for the Hilbertian Sobolev space and the homogeneous Hilbertian Sobolev space, respectively:

$$W^{s,2} = \{f \in \mathcal{D}'(\mathbb{R}^n) : (1 + |\cdot|^2)^{\frac{s}{2}} \widehat{f}(\cdot) \in L^2\},$$

$$\dot{W}^{s,2} = \{f \in \mathcal{D}'(\mathbb{R}^n) : D^s f = \mathcal{F}^{-1}\left(|\cdot|^s \widehat{f}(\cdot)\right) \in L^2\}.$$

The expression $|\xi - \tau| \sim 2^{-k}|\eta|$ means that $2^{-k-1}|\eta| \le |\xi - \tau| \le 2^{-k+1}|\eta|$. In this work χ denotes the characteristic function of the set $\{\eta \in \mathbb{R}^2 : |\eta| > 10\}$. The letter M denotes the Hardy-Littlewood maximal operator. We denote the one-dimensional Hausdorff measure in $\mathbb{R}^2$ by σ. Let $\eta, \xi \in \mathbb{R}^2 \setminus \{0\}$. We write

$$\Gamma(\eta) := \left\{x \in \mathbb{R}^2 : \left|x - \frac{\eta}{2}\right| = \frac{|\eta|}{2}\right\}, \tag{7}$$

referring to the circumference centered at $\frac{\eta}{2}$ and radius $\frac{|\eta|}{2}$ and

$$\Lambda(\xi) := \left\{x \in \mathbb{R}^2 : \xi \cdot (x - \xi) = 0\right\} \tag{8}$$

denotes the line orthogonal to ξ that contains the point ξ.

2. Proof of theorem 1

The cubic term in the Born series for backscattering data is given by

$$\widehat{Q_3(q)}(\xi) := \int_{\mathbb{R}^2} e^{ik\theta \cdot y}(qR_+(k^2))^2(q(\cdot)e^{ik\theta \cdot (\cdot)})(y)dy\,,$$

for any $\xi \in \mathbb{R}^2$, where $\xi = -2k\theta$, that is, $k = \frac{|\xi|}{2}$ and $\theta = -\frac{\xi}{|\xi|}$. From lemma 3.1 in [R2], this term can be characterized by the following:

Proposition 2.1. *For any $\eta \in \mathbb{R}^2 \setminus \{0\}$,*

$$\widehat{Q_3(q)}(\eta) = p.v. \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{[\xi \cdot (\eta - \xi)] [\tau \cdot (\eta - \tau)]}\, d\xi d\tau \tag{9}$$

$$+ 2\frac{i\pi}{|\eta|} p.v. \int_{\mathbb{R}^2} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\tau \cdot (\eta - \tau)}\, d\sigma(\xi)d\tau$$

$$- \frac{\pi^2}{|\eta|^2} \int_{\Gamma(\eta)} \int_{\Gamma(\eta)} \hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)\, d\sigma(\tau)d\sigma(\xi)\,.$$

Notation. For any $\eta \in \mathbb{R}^2 \setminus \{0\}$, we use the following notation:

$$\widehat{Q'(q)}(\eta) := \frac{1}{|\eta|^2} \int_{\Gamma(\eta)} \int_{\Gamma(\eta)} \hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)\, d\sigma(\tau)d\sigma(\xi)\,, \quad \text{and}$$

$$\widehat{Q''(q)}(\eta) := \frac{1}{|\eta|} p.v. \int_{\mathbb{R}^2} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\tau \cdot (\eta - \tau)}\, d\sigma(\xi)d\tau\,. \tag{10}$$

In fact, we are going to prove that

$$Q'(q)\,, Q''(q) \in W^{\alpha,2}(\mathbb{R}^2) + C^{\infty}(\mathbb{R}^2)\,,$$

for any α such that $0 \le \alpha < s + 1$.

2.1. **Estimate of $Q'(q)$.**

Let us split the set $\Gamma(\eta) \times \Gamma(\eta)$ into the two regions:

$$I(\eta) := \left\{ (\xi,\tau) \in \Gamma(\eta) \times \Gamma(\eta) : |\xi - \tau| \ge \frac{|\eta|}{100} \right\} \quad \text{and}$$

$$II(\eta) := \left\{ (\xi,\tau) \in \Gamma(\eta) \times \Gamma(\eta) : |\xi - \tau| < \frac{|\eta|}{100} \right\}\,.$$

In this way, we can write $Q'(q) = Q'_I(q) + Q'_{II}(q)$, where

$$\widehat{Q'_I(q)}(\eta) := \frac{1}{|\eta|^2} \int \int_{I(\eta)} \hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)\, d\sigma(\tau)d\sigma(\xi)\,,$$

for any $\eta \in \mathbb{R}^2 \setminus \{0\}$, and an analogous definition for $\widehat{Q'_{II}(q)}(\eta)$. We will prove that

$$\left\| \mathcal{F}^{-1}\left(\chi\widehat{Q'_{II}(q)} \right) \right\|_{\dot{W}^{\alpha,2}} \le C\|q\|^2_{L^2}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\,, \quad \text{and} \tag{11}$$

$$\left\| \mathcal{F}^{-1}\left(\chi\widehat{Q'_I(q)} \right) \right\|_{\dot{W}^{\alpha,2}} \le C(\varepsilon) \left(\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|^2_{L^2} \right) \|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\,, \tag{12}$$

if $\varepsilon > 0$, $0 < \alpha + \varepsilon < 2$, where $C(\varepsilon)$ is a positive constant depending on ε.

Proof of estimate (11). We know that $II(\eta) \subset II_<(\eta) \cup II_>(\eta)$, where

$$II_<(\eta) := \left\{ (\xi, \tau) \in \Gamma(\eta) \times \Gamma(\eta) : |\xi - \tau| < \frac{|\eta|}{100}, \; |\xi|, |\tau| \leq \left(\frac{\sqrt{2}}{2} + \frac{1}{100} \right) |\eta| \right\}, \quad \text{and}$$

$$II_>(\eta) := \left\{ (\xi, \tau) \in \Gamma(\eta) \times \Gamma(\eta) : |\xi - \tau| < \frac{|\eta|}{100}, \; |\eta - \xi|, |\eta - \tau| \leq \left(\frac{\sqrt{2}}{2} + \frac{1}{100} \right) |\eta| \right\}.$$

By taking the change of variables $\xi' = \eta - \xi$, $\tau' = \eta - \tau$, by Fubini's theorem and the symmetry property $(\xi, \tau) \in II_>(\eta) \Leftrightarrow (\tau, \xi) \in II_>(\eta)$, we have

$$\left| \widehat{Q'_{II}(q)}(\eta) \right| \leq 2 \, \widehat{Q'_{II_>}(q)}(\eta), \tag{13}$$

where $\widehat{Q'_{II_>}(q)}(\eta) := \frac{1}{|\eta|^2} \int \int_{II_>(\eta)} |\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)| \, d\sigma(\tau)d\sigma(\xi)$. Applying the Cauchy-Schwartz inequality to the last expression and by the properties of the region $II_>$ we may write

$$\left\| \mathcal{F}^{-1} \left(\chi \widehat{Q'_{II_>}(q)} \right) \right\|_{\dot{W}^{\alpha,2}}^2 = \int_{\{\eta : |\eta| \geq 10\}} |\eta|^{2\alpha} \left| \widehat{Q'_{II_>}(q)}(\eta) \right|^2 d\eta$$

$$\leq \int_X |\eta|^{2\alpha - 4} \int_{\{\xi \in \Gamma(\eta) : |\xi| \geq C_2 |\eta|\}} |\hat{q}(\xi)|^2 \int_{\Gamma(\eta)} |\hat{q}(\eta - \tau)|^2 d\sigma(\tau) \, d\sigma(\xi) \, F(\eta) \, d\eta, \tag{14}$$

where $X := \{\eta \in \mathbb{R}^2 : |\eta| > 10\}$, $F(\eta) = \int \int_{\Gamma(\eta) \times \Gamma(\eta)} |\hat{q}(\tau' - \xi')|^2 d\sigma(\tau')d\sigma(\xi')$ and $C_2 := \left(1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{100} \right)^2 \right)^{\frac{1}{2}}$. We know that

$$\{(\eta, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\eta| > 10, \, \xi \in \Gamma(\eta), \, |\xi| \geq C_2 |\eta|\}$$

$$\subset \{(\eta, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi| > 1, \, |\eta| \leq C_2^{-1} |\xi|, \, \eta \in \Lambda(\xi)\},$$

and by lemma 4.1, we may change the order of integration and estimate expression (14) by

$$\leq \int_{\{\xi \in \mathbb{R}^2 : |\xi| > 1\}} \frac{|\hat{q}(\xi)|^2}{|\xi|} \int_{\Omega(\xi)} |\eta|^{2\alpha - 3} \int_{\Gamma(\eta)} |\hat{q}(\eta - \tau)|^2 d\sigma(\tau) \, F(\eta) \, d\sigma(\eta) \, d\xi$$

$$\leq C \|q\|_{L^2}^2 \int_{\{\xi \in \mathbb{R}^2 : |\xi| > 1\}} \frac{|\hat{q}(\xi)|^2}{|\xi|} \int_{\Omega(\xi)} |\eta|^{2\alpha - 2} \int_{\Gamma(\eta)} |\hat{q}(\eta - \tau)|^2 d\sigma(\tau) \, d\sigma(\eta) \, d\xi, \tag{15}$$

where $\Omega(\xi) := \{\eta \in \Lambda(\xi) : |\eta| \leq C_2^{-1} |\xi|\}$, and we have used the inequality $F(\eta) \leq C|\eta|\|q\|_{L^2}^2$. Let us see the last inequality. If we widen the curve $\Gamma(\eta)$ until $\Gamma_1(\eta) := \left\{ \tau \in \mathbb{R}^2 : \left| \left| \tau - \frac{\eta}{2} \right| - \frac{|\eta|}{2} \right| < 1 \right\}$, by part 1) of lemma 4.2 we have:

$$F(\eta) \leq C \int_{\Gamma(\eta)} \int_{\Gamma_1(\eta)} M\hat{q}(\tau - \xi')^2 d\tau d\sigma(\xi') \leq C \|M\hat{q}\|_{L^2}^2 \, \sigma(\Gamma(\eta)) \leq C|\eta|\|q\|_{L^2}^2,$$

where the last inequality follows from the boundedness of Hardy-Littlewood maximal operator in $L^2(\mathbb{R}^n)$ and Plancherel identity, since the measure of $\Gamma(\eta)$ is $\pi|\eta|$. In the same way,

$$\int_{\Gamma(\eta)} |\hat{q}(\eta - \tau)|^2 d\sigma(\tau) \leq C \int_{\Gamma_1(\eta)} \left| M\hat{q}(\eta - \tau') \right|^2 d\tau' \leq C \int_{\mathbb{R}^2} \left| M\hat{q}(x) \right|^2 dx \leq C\|q\|_{L^2}^2.$$

Obviously, if $\eta \in \Omega(\xi)$ then $|\xi| \sim |\eta|$. Expression (15) can be bounded by

$$C \|q\|_{L^2}^4 \int_{\{\xi \in \mathbb{R}^2 : |\xi| > 1\}} \frac{|\hat{q}(\xi)|^2}{|\xi|} |\xi|^{2\alpha - 2} \sigma\left(\Omega(\xi)\right) d\xi \leq C\|q\|_{L^2}^4 \|q\|_{\dot{W}^{\alpha - 1 + \varepsilon, 2}}^2.$$

So we have proved estimate (11).

$\square$

Proof of estimate (12). Taking the change of variable $\tau = \eta - \tau'$, we have

$$\widehat{Q'_I(q)}(\eta) = \frac{1}{|\eta|^2} \int \int_{I(\eta)} \hat{q}(\xi)\hat{q}(\eta - \tau')\hat{q}(\tau' - \xi)\, d\sigma(\tau')d\sigma(\xi)$$

$$= \frac{1}{|\eta|^2} \int \int_{\{(\xi,\tau)\in \Gamma(\eta)\times\Gamma(\eta):|\xi - (\eta-\tau)|\geq \frac{|\eta|}{100}\}} \hat{q}(\xi)\hat{q}(\tau)\hat{q}(\eta - \tau - \xi)\, d\sigma(\tau)d\sigma(\xi)\,.$$

Notice that if $|\eta| \geq 4$ we can write

$$\{(\xi,\tau)\in \Gamma(\eta)\times\Gamma(\eta): |\xi - (\eta-\tau)| \geq \frac{|\eta|}{100}\} = \bigcup_{k=2}^{[\log_2|\eta|]} I_k(\eta) \cup I_0(\eta) \cup I_\infty(\eta)\,,$$

where for any $k \in \mathbb{Z}$ such that $2 \leq k \leq [\log_2|\eta|]$ we denote:

$$I_k(\eta) := \left\{(\xi,\tau)\in \Gamma(\eta)\times\Gamma(\eta): |\xi - \tau| \sim 2^{-k}|\eta|,\ |\xi - (\eta-\tau)| \geq \frac{|\eta|}{100}\right\}, \qquad (16)$$

$$I_0(\eta) := \left\{(\xi,\tau)\in \Gamma(\eta)\times\Gamma(\eta): |\xi - \tau| \geq \frac{|\eta|}{2},\ |\xi - (\eta-\tau)| \geq \frac{|\eta|}{100}\right\},$$

$$I_\infty(\eta) := \left\{(\xi,\tau)\in \Gamma(\eta)\times\Gamma(\eta): |\xi - \tau| \leq 1,\ |\xi - (\eta-\tau)| \geq \frac{|\eta|}{100}\right\}.$$

Note that cases $k = 0$ and $k = \infty$ are needed since the union from $k = 2$ to $k = [\log_2|\eta|]$ only covers the set of ξ, τ such that $1 \leq |\xi - \tau| \leq \frac{|\eta|}{2}$.

For each $k \in \{2,3,...\}$ we define

$$\widehat{Q'_{I_k}(q)}(\eta) := \chi_{\{\eta:|\eta|\geq 2^k\}}(\eta) \frac{1}{|\eta|^2} \int \int_{I_k(\eta)} |\hat{q}(\xi)\hat{q}(\tau)\hat{q}(\eta-\tau-\xi)|\, d\sigma(\tau)d\sigma(\xi)\,,$$

and the same expression for $I_0(\eta)$, $I_\infty(\eta)$, but with $\chi_{\{\eta:|\eta|\geq 10\}}(\eta)$. For any $|\eta| \geq 10$,

$$\left|\widehat{Q'_I(q)}(\eta)\right| \leq \sum_{k=2}^{+\infty} \left|\widehat{Q'_{I_k}(q)}(\eta)\right| + \left|\widehat{Q'_{I_0}(q)}(\eta)\right| + \left|\widehat{Q'_{I_\infty}(q)}(\eta)\right|\,,$$

and then to prove (12) we use

$$\left\|\mathcal{F}^{-1}\left(\chi\widehat{Q'_I(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq \sum_{k=2}^{+\infty} \|Q'_{I_k}(q)\|_{\dot{W}^{\alpha,2}} + \|Q'_{I_0}(q)\|_{\dot{W}^{\alpha,2}} + \|Q'_{I_\infty}(q)\|_{\dot{W}^{\alpha,2}}\,.$$

Let ε, α be real numbers with $\varepsilon > 0$. For each $k \in \{2,3,...\}$ we claim

$$\|Q'_{I_k}(q)\|_{\dot{W}^{\alpha,2}} \leq C \cdot 2^{-\varepsilon k} \|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\,. \qquad (17)$$

Assume $0 < \alpha + \varepsilon < 2$. Then we claim

$$\|Q'_{I_\infty}(q)\|_{\dot{W}^{\alpha,2}} \leq C\|q\|_{L^2}\left(\|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{L^2}\|q\|_{\dot{W}^{\alpha-\frac{3}{2}+\varepsilon,2}}\right) \text{ and} \qquad (18)$$

$$\|Q'_{I_0}(q)\|_{\dot{W}^{\alpha,2}} \leq C\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\,. \qquad (19)$$

In the following, we use the notation in lemma 4.4, which is the key of the proof of the above claims.

Proof of claim (17). We take $I_k(\eta) = I_k^1(\eta) \cup I_k^2(\eta)$, where $I_k^1(\eta) := \left\{ (\xi, \tau) \in I_k(\eta) : |\tau - \eta| \geq \frac{2^{-k}|\eta|}{100} \right\}$ and

$$I_k^2(\eta) := \left\{ (\xi, \tau) \in I_k(\eta) : |\tau - \eta| \leq \frac{2^{-k}|\eta|}{100} \right\}.$$

For each $j \in \{1, 2\}$, let us define $\widehat{Q'_{I_k^j}(q)}(\eta)$ in the obvious way multiplying by $\chi_{\{\eta : |\eta| \geq 2^k\}}(\eta)$. By Cauchy-Schwartz inequality, and for $|\eta| \geq 2^k$, $j \in \{1, 2\}$:

$$\widehat{Q'_{I_k^j}(q)}(\eta) \leq \frac{1}{|\eta|^2} \left(\int \int_{I_k^j(\eta)} |\hat{q}(\xi)\hat{q}(\tau)|^2 d\sigma(\tau) d\sigma(\xi) \int \int_{I_k^j(\eta)} |\hat{q}(\eta - \tau' - \xi')|^2 d\sigma(\tau') d\sigma(\xi') \right)^{\frac{1}{2}}. \quad (20)$$

Let us begin with $Q'_{I_k^1}(q)$. By lemma 4.2, we have

$$\int \int_{I_k^1(\eta)} |\hat{q}(\xi)\hat{q}(\tau)|^2 d\sigma(\tau) d\sigma(\xi) \leq C \|q\|_{L^2}^2 \int_{\Psi_k(\eta)} |\hat{q}(\tau)|^2 d\sigma(\tau),$$

where $\Psi_k(\eta) := \left\{ \tau \in \Gamma(\eta) : |\eta - \tau| \geq \frac{2^{-k}|\eta|}{100} \right\}$. Since $|\eta|^{-2\varepsilon} \leq 2^{-2\varepsilon k}$ and by lemma 4.1,

$$\|Q'_{I_k^1}(q)\|_{\dot{W}^{\alpha,2}}^2$$

$$\leq C \cdot 2^{-2\varepsilon k} \|q\|_{L^2}^2 \int_{\mathbb{R}^2} |\eta|^{2\alpha - 4 + 2\varepsilon} \int_{\Psi_k(\eta)} |\hat{q}(\tau)|^2 d\sigma(\tau) \int \int_{I_k(\eta)} |\hat{q}(\eta - \tau' - \xi')|^2 d\sigma(\tau') d\sigma(\xi') d\eta$$

$$= C \cdot 2^{-2\varepsilon k} \|q\|_{L^2}^2 \int_{\mathbb{R}^2} \frac{|\hat{q}(\tau)|^2}{|\tau|} \int_{\Omega_k(\tau)} |\eta|^{2\alpha - 3 + 2\varepsilon} \int \int_{I_k(\eta)} |\hat{q}(\eta - \tau' - \xi')|^2 d\sigma(\tau') d\sigma(\xi') d\sigma(\eta) d\tau$$

$$= C \cdot 2^{-2\varepsilon k} \|q\|_{L^2}^2 \int_{\mathbb{R}^2} \frac{|\hat{q}(\tau)|^2}{|\tau|} F_k(\tau) \, d\tau \leq C \cdot 2^{-2\varepsilon k} \|q\|_{L^2}^2 \|q\|_{\dot{W}^{-\frac{1}{2},2}}^2 \|q\|_{\dot{W}^{\alpha - 1 + \varepsilon, 2}}^2,$$

where the last inequality follows from part $i)$ of lemma 4.4 with $C_1 = \frac{1}{100}$ and $F_k(\tau), \Omega_k(\tau)$ are defined in (4.1), (4.4).

We can bound the term $\widehat{Q'_{I_k^2}(q)}(\eta)$ in a similar way. Firstly, we estimate the factor

$$\int \int_{I_k^2(\eta)} |\hat{q}(\xi)\hat{q}(\tau)|^2 d\sigma(\tau) d\sigma(\xi)$$

by $C \|q\|_{L^2}^2 \int_{\widetilde{\Psi}_k(\eta)} |\hat{q}(\xi)|^2 d\sigma(\xi)$, where $\widetilde{\Psi}_k(\eta) := \{ \xi \in \Gamma(\eta) : |\eta - \xi| \geq \frac{49}{100} 2^{-k}|\eta| \}$, by using lema 4.2 as above. In order to estimate the expression $\|Q'_{I_k^2}(q)\|_{\dot{W}^{\alpha,2}}^2$ as before, we proceed similarly so that the variable ξ now acts just as the variable τ before, obtaining that

$$\|Q'_{I_k^2}(q)\|_{\dot{W}^{\alpha,2}}^2 \leq C \, 2^{-2\varepsilon k} \|q\|_{L^2}^2 \int_{\mathbb{R}^2} \frac{|\hat{q}(\xi)|^2}{|\xi|} F_k(\xi) d\xi \leq C \cdot 2^{-2\varepsilon k} \|q\|_{L^2}^2 \|q\|_{\dot{W}^{-\frac{1}{2},2}}^2 \|q\|_{\dot{W}^{\alpha - 1 + \varepsilon, 2}}^2,$$

where the last inequality follows from part $i)$ of lemma 4.4 with $C_1 = \frac{49}{100}$.

$\square$

Proof of claim (18). We split the set $I_\infty(\eta)$ into $I_\infty^1(\eta)$ (where $|\tau| \geq 1$) and $I_\infty^2(\eta)$ (where $|\tau| \leq 1$). Let us denote $\widehat{Q'_{I_\infty^j}(q)}(\eta)$ in the obvious way, for $j = 1, 2$. In order to estimate the Sobolev norm of $Q'_{I_\infty^j}(q)$, $j = 1, 2$, we apply Cauchy-Schwartz inequality as in (20), by lemma 4.2, for $I_\infty^1(\eta)$, we may do $|\hat{q}(\eta - \xi' - \tau')| \leq CM\hat{q}(\eta - 2\tau')$, since $|\xi' - \tau'| \leq 1$, and for $I_\infty^2(\eta)$, $|\hat{q}(\eta - \xi' - \tau')| \leq CM\hat{q}(\eta - \tau')$, since $|\xi'| \leq 2$, we bound the integral involving $\hat{q}(\xi)$ by lemma 4.2, leading for $I_\infty^1(\eta)$ to $\int \int_{I_\infty^1(\eta)} M\hat{q}(\eta - 2\tau')^2 d\sigma(\xi') d\sigma(\tau') \leq C \int_{\Gamma(\eta)} M\hat{q}(\eta - 2\tau')^2 d\sigma(\tau')$ (the same with

$M\hat{q}(\eta - \tau')$, for $I_\infty^2(\eta)$), change the order of integration in τ, η by lemma 4.1, and finally, by parts $ii)$ and $iii)$ of lemma 4.4 (provided that $0 < \alpha + \varepsilon < 2$) we get

$$\|Q'_{I_\infty^1}(q)\|_{\dot{W}^{\alpha,2}}^2 \leq C\|q\|_{L^2}^2 \left(\|q\|_{\dot{W}^{-\frac{1}{2},2}}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}^2 + \|q\|_{L^2}^2\|q\|_{\dot{W}^{\alpha-\frac{3}{2}+\varepsilon,2}}^2\right) \quad \text{and}$$

$$\|Q'_{I_\infty^2}(q)\|_{\dot{W}^{\alpha,2}}^2 \leq C\|q\|_{L^2}^2\|q\|_{\dot{W}^{-\frac{1}{2},2}}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}^2\,,$$

respectively.

$\square$

Proof of claim (19). We also split the set $I_0(\eta)$ into $I_0^1(\eta)$ (where $|\eta - \tau| \geq \frac{|\eta|}{4}$) and $I_0^2(\eta)$ (where $|\eta - \tau| < \frac{|\eta|}{4}$). Note that on the region $I_0^2(\eta)$, $|\eta - \xi| \geq \frac{|\eta|}{4}$ holds. In both cases we apply Cauchy-Schwartz inequality in the same way as in the previous cases, bound $|\hat{q}(\xi)|^2$ (for $I_0^1(\eta)$) or $|\hat{q}(\tau)|^2$ (for $I_0^2(\eta)$) by the maximal operator by lemma 4.2, change the order of integration in the variables τ, η, for $I_0^1(\eta)$ (in the variables ξ, η, for $I_0^2(\eta)$) by lemma 4.1 and finally, by part $i)$ of lemma 4.4, with $k = 1$ and $C_1 = \frac{1}{2}$, we get

$$\|Q'_{I_0^j}(q)\|_{\dot{W}^{\alpha,2}}^2 \leq C\|q\|_{L^2}^2\|q\|_{\dot{W}^{-\frac{1}{2},2}}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}^2\,, \quad j = 1, 2\,.$$

$\square$

Hence by estimates (17), (18) and (19) we can write

$$\left\|\mathcal{F}^{-1}\left(\chi\widehat{Q'_I(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq \frac{C \cdot 2^{-\varepsilon}}{2^\varepsilon - 1}\|q\|_{L^2}\left(\|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{L^2}\|q\|_{\dot{W}^{\alpha-\frac{3}{2}+\varepsilon,2}}\right)\,,$$

and we have proved (12).

$\square$

To obtain the non-homogeneous Sobolev norm we proceed as follows. By lemma 4.2, $q \in W^{-(1-\varepsilon),2}(\mathbb{R}^2)$ holds for $0 < \varepsilon < 1$, and replacing α by 0 in (11) we get that

$$\mathcal{F}^{-1}\left(\chi\widehat{Q'_{II}(q)}\right) \in L^2(\mathbb{R}^2)\,.$$

Note that estimates (17), (18) and (19) remain true if $\alpha = 0$ (assuming that $0 < \varepsilon < 2$ to guarantee the estimate (18)). Hence $\mathcal{F}^{-1}\left(\chi\widehat{Q'_I(q)}\right) \in L^2(\mathbb{R}^2)$. It follows that

$$Q'(q) = \mathcal{F}^{-1}\left((1-\chi)\widehat{Q'(q)}\right) + \mathcal{F}^{-1}\left(\chi\widehat{Q'(q)}\right)\,,$$

where the first term is a function belonging to the class $C^\infty(\mathbb{R}^2)$, and the second one is in $W^{\alpha,2}(\mathbb{R}^2)$ if $0 \leq \alpha \leq s+1-\varepsilon$ (with $0 < \varepsilon < 2$ arbitrary, provided that $s < 1$), that is to say, if $0 \leq \alpha < s+1$. So, we have finished with the term $Q'(q)$.

2.2. **Estimate of $Q''(q)$.**

The singularities of the integral (10) are those points τ in the plane such that $\tau \cdot (\eta - \tau) = 0$, that is, the set $\Gamma(\eta)$. So, we decompose the plane in an annulus containing $\Gamma(\eta)$ and its complement.

Next, we decompose the first annulus in diadic coronas and try to treat the corresponding integral terms. Let

$$N := \max\{[\log_2 |\eta|] - 2, 1\} = \begin{cases} [\log_2 |\eta|] - 2, & \text{if } |\eta| \geq 16, \\ 1, & \text{if } |\eta| < 16. \end{cases}$$

Let j_0 be the lowest integer such that $j_0 \geq -1 - \log_2(\delta_0)$, with δ_0 from lemma 4.5 (see the appendix). We define the sets

$$\Gamma_{j_0^-}(\eta) := \{\tau \in \mathbb{R}^2 : \left|\left|\tau - \frac{\eta}{2}\right| - \frac{|\eta|}{2}\right| > 2^{-j_0-1}|\eta|\},$$

$$\Gamma_j(\eta) := \{\tau \in \mathbb{R}^2 : 2^{-j-2}|\eta| < \left|\left|\tau - \frac{\eta}{2}\right| - \frac{|\eta|}{2}\right| \leq 2^{-j-1}|\eta|\},$$

$$\Gamma_\infty(\eta) := \{\tau \in \mathbb{R}^2 : \left|\left|\tau - \frac{\eta}{2}\right| - \frac{|\eta|}{2}\right| \leq 2\}, \tag{21}$$

with $j_0 \leq j \leq N$. If $j \geq j_0$ it is true that $j \leq N \Leftrightarrow |\eta| \geq 2^{j+2}$ (for $|\eta| \geq 16$). So, we also define for $j_0 \leq j < \infty$:

$$\widehat{Q_j''(q)}(\eta) := \widetilde{\chi}(|\eta|)\frac{1}{|\eta|} \int_{\Gamma_j(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\tau \cdot (\eta - \tau)} \, d\sigma(\xi)d\tau, \tag{22}$$

where $\widetilde{\chi} = \chi_{[2^{j+2}, +\infty)}$, and the obvious notations for $\widehat{Q_\infty''(q)}(\eta)$, $\widehat{Q_{j_0^-}''(q)}(\eta)$ without the characteristic function. Since $\mathbb{R}^2 = \bigcup_{j=j_0}^N \Gamma_j(\eta) \cup \Gamma_\infty(\eta) \cup \Gamma_{j_0^-}(\eta)$, for any $\eta \in \mathbb{R}^2 \setminus \{0\}$, it follows that

$$\widehat{Q''(q)}(\eta) = \widehat{Q_{j_0^-}''(q)}(\eta) + \sum_{j=j_0}^N \widehat{Q_j''(q)}(\eta) + \widehat{Q_\infty''(q)}(\eta) = \widehat{Q_{j_0^-}''(q)}(\eta) + \sum_{j=j_0}^\infty \widehat{Q_j''(q)}(\eta) + \widehat{Q_\infty''(q)}(\eta).$$

It is easy to see that

$$\left\|\mathcal{F}^{-1}\left(\chi \widehat{Q_{j_0^-}''(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq C\|q\|_{L^2}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}. \tag{23}$$

Bound of the corona terms.

By Minkowski's inequality, we have

$$\left\|\mathcal{F}^{-1}\left(\chi \sum_{j=j_0}^\infty \widehat{Q_j''(q)}\right)\right\|_{\dot{W}^{\alpha,2}} = \left\|\sum_{j=j_0}^\infty \mathcal{F}^{-1}\left(\chi\widehat{Q_j''(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq \sum_{j=j_0}^\infty \left\|\mathcal{F}^{-1}\left(\chi\widehat{Q_j''(q)}\right)\right\|_{\dot{W}^{\alpha,2}}.$$

If $j \geq j_0$ and $\tau \in \Gamma_j(\eta)$, $|\tau \cdot (\eta - \tau)| \geq 2^{-j-3}|\eta|^2$, from where we deduce that

$$|\widehat{Q_j''(q)}(\eta)| \leq 2^{j+3}\chi_{(2^{j+1}, +\infty)}\left(\frac{|\eta|}{2}\right)\frac{1}{|\eta|^3} \int_{\Gamma_j(\eta)} \int_{\Gamma(\eta)} |\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)| \, d\sigma(\xi)d\tau.$$

It follows that

$$\Gamma_j(\eta) \subset \{\tau \in \mathbb{R}^2 : \left||\tau - \frac{\eta}{2}| - |\frac{\eta}{2}|\right| < 2^{-j-1}|\eta|\},$$

hence, applying the key lemma 4.5 with $\delta = 2^{-j-1}$, we know that there exist $\delta_0 > 0$, $\beta > 1$ and $C > 0$ so that for any $j \in \mathbb{N}$ satisfying $2^{-j-1} < \delta_0$ (that is, $j \geq j_0$), we have that

$$\left\|\mathcal{F}^{-1}\left(\chi\widehat{Q_j''(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq C2^{j+3}(2^{-j-1})^\beta\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\left(\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|_{L^2}^2\right),$$

where $\varepsilon > 0$ satisfies $0 < \alpha + \varepsilon < 2$. We can write

$$\sum_{j=j_0}^{\infty} \left\|\mathcal{F}^{-1}\left(\chi \widehat{Q_j''(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq C \sum_{j=j_0}^{\infty} 2^{-j(\beta-1)} 2^{3-\beta} \|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} \left(\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|_{L^2}^2\right)$$

$$= C\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} \left(\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|_{L^2}^2\right). \tag{24}$$

The series $\sum_{j=j_0}^{\infty} 2^{-j(\beta-1)}$ converges because $\beta > 1$.

Bound of the singular part close to $\Gamma(\eta)$.

We are going to prove the estimate:

$$\|\mathcal{F}^{-1}(\chi \widehat{Q_\infty''(q)})\|_{\dot{W}^{\alpha,2}} \leq C \left[\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|_{L^2}^2\right] \|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}, \tag{25}$$

for a constant $C > 0$ depending on the support of q, and provided that $0 < \alpha + \varepsilon < 2$. Up to now, we have avoided the singular region $\Gamma(\eta)$. The domain $\Gamma_\infty(\eta)$ contains it. In order to calculate the principal value of the integral on $\Gamma_\infty(\eta)$, we integrate on two rings whose radial distance to the singular circumference is $\varepsilon > 0$ and pass to limit when $\varepsilon \to 0^+$. We write:

$$\widehat{Q_\infty''(q)}(\eta) = \frac{1}{|\eta|} \lim_{\varepsilon \to 0^+} \left(\int_{\Gamma_\varepsilon^+(\eta)} + \int_{\Gamma_\varepsilon^-(\eta)}\right) \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau)\hat{q}(\tau-\xi)}{\tau \cdot (\eta-\tau)} d\sigma(\xi)d\tau, \tag{26}$$

where

$$\Gamma_\varepsilon^+(\eta) := \{\tau \in \mathbb{R}^2 : \varepsilon < \left|\tau - \frac{\eta}{2}\right| - \frac{|\eta|}{2} < 2\} \text{ and}$$

$$\Gamma_\varepsilon^-(\eta) := \{\tau \in \mathbb{R}^2 : \varepsilon < \frac{|\eta|}{2} - \left|\tau - \frac{\eta}{2}\right| < 2\}.$$

Let us take the change of variables $\tau' = \phi(\tau)$, $\tau \in \Gamma_\varepsilon^-(\eta)$, that sends τ to its symmetrical point $\tau' \in \Gamma_\varepsilon^+(\eta)$ with respect to $\Gamma(\eta)$ on the radial direction with centre at $\frac{\eta}{2}$. We have

$$\tau' = \eta - \tau + |\eta| \frac{\tau - \frac{\eta}{2}}{\left|\tau - \frac{\eta}{2}\right|}.$$

A straightforward calculation leads up to the following identities:

$$\left|\phi(\tau) - \frac{\eta}{2}\right| - \frac{|\eta|}{2} = -\left(\left|\tau - \frac{\eta}{2}\right| - \frac{|\eta|}{2}\right), \tag{27}$$

$$|D\phi(\tau)| = 1 + 2 \frac{\frac{|\eta|}{2} - \left|\tau - \frac{\eta}{2}\right|}{\left|\tau - \frac{\eta}{2}\right|}, \tag{28}$$

$$|\phi(\tau) - \tau| = 2\left(\frac{|\eta|}{2} - \left|\tau - \frac{\eta}{2}\right|\right), \tag{29}$$

$$\phi(\tau) \cdot (\eta - \phi(\tau)) = \left(\frac{|\eta|}{2} + \left|\phi(\tau) - \frac{\eta}{2}\right|\right) \cdot \left(\left|\tau - \frac{\eta}{2}\right| - \frac{|\eta|}{2}\right),$$

$$\tau \cdot (\eta - \tau) = \left(\frac{|\eta|}{2} + \left|\tau - \frac{\eta}{2}\right|\right) \cdot \left(\frac{|\eta|}{2} - \left|\tau - \frac{\eta}{2}\right|\right). \tag{30}$$

Taking the change $\tau' = \phi(\tau)$ in the first integral in (26), we get

$$\int_{\Gamma_\varepsilon^+(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta-\tau')\hat{q}(\tau'-\xi)}{\tau' \cdot (\eta-\tau')} d\sigma(\xi)d\tau'$$

$$= \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \phi(\tau))\hat{q}(\phi(\tau) - \xi)}{\phi(\tau) \cdot (\eta - \phi(\tau))} |D\phi(\tau)| \, d\sigma(\xi) d\tau \, .$$

Then we have $\widehat{Q''_\infty(q)}(\eta)$

$$= \lim_{\varepsilon \to 0^+} |\eta|^{-1} \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \left[\frac{\hat{q}(\xi)\hat{q}(\eta - \phi(\tau))\hat{q}(\phi(\tau) - \xi)}{\phi(\tau) \cdot (\eta - \phi(\tau))} |D\phi(\tau)| + \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\tau \cdot (\eta - \tau)} \right] d\sigma(\xi) d\tau$$

$$= \lim_{\varepsilon \to 0^+} |\eta|^{-1} \Bigg[\int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \phi(\tau))\hat{q}(\phi(\tau) - \xi) - \hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{|\frac{\eta}{2}|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2} |D\phi(\tau)| \, d\sigma(\xi) d\tau$$

$$+ \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{|\frac{\eta}{2}|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2} |D\phi(\tau)| \, d\sigma(\xi) d\tau + \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\tau \cdot (\eta - \tau)} \, d\sigma(\xi) d\tau \Bigg]$$

$$= \lim_{\varepsilon \to 0^+} |\eta|^{-1} \Bigg[\int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi) \left[\hat{q}(\eta - \phi(\tau))\hat{q}(\phi(\tau) - \xi) - \hat{q}(\eta - \tau)\hat{q}(\tau - \xi) \right]}{|\frac{\eta}{2}|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2} |D\phi(\tau)| \, d\sigma(\xi) d\tau$$

$$- 2 \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\left(|\frac{\eta}{2}| + \left|\phi(\tau) - \frac{\eta}{2}\right| \right) \cdot \left|\tau - \frac{\eta}{2}\right|} \, d\sigma(\xi) d\tau$$

$$+ 2 \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)}{\left(|\frac{\eta}{2}| + |\phi(\tau) - \frac{\eta}{2}| \right) \left(\frac{|\eta|}{2} + \left|\tau - \frac{\eta}{2}\right| \right)} \, d\sigma(\xi) d\tau \Bigg]$$

$$=: \lim_{\varepsilon \to 0^+} \left(I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon \right),$$

where we have to keep identities (27), (28) and (30) in mind and also

$$\frac{1}{|\frac{\eta}{2}|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2} + \frac{1}{\tau \cdot (\eta - \tau)} = \frac{2}{\left(|\frac{\eta}{2}| + |\phi(\tau) - \frac{\eta}{2}| \right) \left(\frac{|\eta|}{2} + \left|\tau - \frac{\eta}{2}\right| \right)} \, .$$

For $|\eta| > 10$, the terms I_2^ε, I_3^ε may be upper bounded by a term like

$$\widehat{J(q)}(\eta) := \chi(\eta) \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{|\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)|}{|\eta|^3} \, d\sigma(\xi) d\tau. \tag{31}$$

If one replaces the characteristic function χ of the set $\{\eta \in \mathbb{R}^2 : |\eta| > 10\}$ by the characteristic function of the complement of a bigger ball our proof for theorem 1 remains valid. In (31) if we replace χ by the characteristic function of the set $\{\eta \in \mathbb{R}^2 : |\eta| > r\}$, with $r \in \mathbb{R}$ such that $r > \frac{2}{\delta_0}$ (for δ_0 from lemma 4.5) then $\widehat{J(q)}(\eta) \leq \widehat{Q''_{\frac{2}{r}}(q)}(\eta)$ holds, according to the notation from lemma 4.5 (since $2 = \frac{2}{|\eta|} |\eta| < \frac{2}{r} |\eta|$ and $\Gamma_\infty(\eta) \subset \Gamma_{\frac{2}{r}}(\eta)$), that is, we may apply lemma 4.5 with $\delta = \frac{2}{r} (< \delta_0)$ and get that there exists a constant $C > 0$ such that

$$\|J(q)\|_{\dot{W}^{\alpha,2}} \leq C \left(\|q\|_{L^2} \|q\|_{\dot{W}^{-\frac{1}{2},2}} \|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{L^2}^2 \|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} \right). \tag{32}$$

On the one hand,

$$I_1^\varepsilon = \frac{1}{|\eta|} \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi) \left[\hat{q}(\eta - \phi(\tau))\hat{q}(\phi(\tau) - \xi) - \hat{q}(\eta - \tau)\hat{q}(\tau - \xi) \right]}{|\frac{\eta}{2}|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2} \, d\sigma(\xi) d\tau$$

$$+ \frac{2}{|\eta|} \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{\hat{q}(\xi) \left[\hat{q}(\eta - \phi(\tau))\hat{q}(\phi(\tau) - \xi) - \hat{q}(\eta - \tau)\hat{q}(\tau - \xi) \right]}{|\frac{\eta}{2}|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2} \frac{\frac{|\eta|}{2} - \left|\tau - \frac{\eta}{2}\right|}{\left|\tau - \frac{\eta}{2}\right|} \, d\sigma(\xi) d\tau \, .$$

If $\tau \in \Gamma_\varepsilon^-(\eta)$ and $|\eta| > 10$,

$$0 < \frac{\frac{|\eta|}{2} - \left|\tau - \frac{\eta}{2}\right|}{\left|\tau - \frac{\eta}{2}\right|} < 1.$$

That is,

$$|I_1^\varepsilon| \le \frac{3}{|\eta|} \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{|\hat{q}(\xi)\hat{q}(\phi(\tau) - \xi)\left[\hat{q}(\eta - \phi(\tau)) - \hat{q}(\eta - \tau)\right]|}{\left|\left|\frac{\eta}{2}\right|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2\right|} \, d\sigma(\xi) d\tau$$

$$+ \frac{3}{|\eta|} \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{|\hat{q}(\xi)\hat{q}(\eta - \tau)\left[\hat{q}(\phi(\tau) - \xi) - \hat{q}(\tau - \xi)\right]|}{\left|\left|\frac{\eta}{2}\right|^2 - \left|\phi(\tau) - \frac{\eta}{2}\right|^2\right|} \, d\sigma(\xi) d\tau$$

$$=: \widehat{J_1(q)}(\eta) + \widehat{J_2(q)}(\eta).$$

The term $\widehat{J_1(q)}(\eta)$ may be bounded by Calderón estimate (see Section 2 in [H]):

$$|f(x) - f(y)| \le C \left(M(\nabla f)(x) + M(\nabla f)(y)\right)|x - y| \quad \text{a.e.},$$

provided that $f \in \dot{W}^{1,p}(\mathbb{R}^n) := \{g \in \mathcal{D}'(\mathbb{R}^n) \,/\, \nabla g \in L^p(\mathbb{R}^n)\}$, for some $p > 1$. So, by (29) we attain that

$$|\widehat{J_1(q)}(\eta)| \le C \frac{1}{|\eta|} \int_{\Gamma_\varepsilon^-(\eta)} \int_{\Gamma(\eta)} \frac{[M(\nabla\hat{q})(\eta - \phi(\tau)) + M(\nabla\hat{q})(\eta - \tau)] |\hat{q}(\xi)\hat{q}(\phi(\tau) - \xi)|}{\left|\phi(\tau) - \frac{\eta}{2}\right| + |\frac{\eta}{2}|} \, d\sigma(\xi) d\tau$$

$$=: C \left(\widehat{J_1^1(q)}(\eta) + \widehat{J_1^2(q)}(\eta)\right).$$

Let $\hat{f} := M(\nabla\hat{q})$. It holds

$$\widehat{J_1^1(q)}(\eta) = \frac{1}{|\eta|} \int_{\Gamma_\varepsilon^+(\eta)} \int_{\Gamma(\eta)} \frac{\hat{f}(\eta - \tau') |\hat{q}(\xi)\hat{q}(\tau' - \xi)|}{\left|\tau' - \frac{\eta}{2}\right| + |\frac{\eta}{2}|} |D\phi^{-1}(\tau')| \, d\sigma(\xi) d\tau'$$

$$\le \frac{1}{|\eta|} \left(\int_{\Gamma_\varepsilon^+(\eta)} \int_{\Gamma(\eta)} \frac{\hat{f}(\eta - \tau') |\hat{q}(\xi)\hat{q}(\tau' - \xi)|}{\left|\tau' - \frac{\eta}{2}\right| + |\frac{\eta}{2}|} \, d\sigma(\xi) d\tau' \right. \tag{33}$$

$$\left. + 2 \int_{\Gamma_\varepsilon^+(\eta)} \int_{\Gamma(\eta)} \frac{\hat{f}(\eta - \tau') |\hat{q}(\xi)\hat{q}(\tau' - \xi)|}{\left|\tau' - \frac{\eta}{2}\right| + |\frac{\eta}{2}|} \frac{\left|\tau' - \frac{\eta}{2}\right| - \frac{|\eta|}{2}}{\left|\tau' - \frac{\eta}{2}\right|} \, d\sigma(\xi) d\tau' \right). \tag{34}$$

The second integral (34) is bounded by

$$\widehat{K(q)}(\eta) := \int_{\Gamma_\varepsilon^+(\eta)} \int_{\Gamma(\eta)} \frac{\hat{f}(\eta - \tau')|\hat{q}(\xi)\hat{q}(\tau' - \xi)|}{|\eta|^3} \, d\sigma(\xi) d\tau'.$$

Applying remark to lemma 4.5 with $\delta = \frac{2}{r}$ (where r is defined in page 11), and by lemma 4.3, we have

$$\|\mathcal{F}^{-1}\left(\chi \widehat{K(q)}\right)\|_{\dot{W}^{\alpha,2}} \le C\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} \left[\|q\|_{L^2}\|f\|_{L^2} + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|f\|_{L^2} + \|f\|_{L^2}\|q\|_{L^2} \right]$$

$$\le C \left[\|q\|_{L^2}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{L^2}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} \right]. \tag{35}$$

The integral (33) is bounded by a positive constant multiplied by

$$\widehat{K'(q)}(\eta) := \int_{\Gamma_\infty(\eta)} \int_{\Gamma(\eta)} \frac{\hat{f}(\eta - \tau')|\hat{q}(\xi)\hat{q}(\tau' - \xi)|}{|\eta|^2} \, d\sigma(\xi) d\tau'.$$

By lemma 4.6 and lemma 4.3, the bound (35) works for $\|\mathcal{F}^{-1}\left(\chi\, K'(q)\right)\|_{\dot{W}^{\alpha,2}}$ too. So we have

$$\left\|\mathcal{F}^{-1}\left(\chi\, \widehat{J_1^1(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq C\left[\|q\|_{L^2}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{L^2}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\right]. \qquad (36)$$

It holds

$$\widehat{J_1^2(q)}(\eta) \leq C\int_{\Gamma_\varepsilon^-(\eta)}\int_{\Gamma(\eta)} \frac{M(\nabla\hat{q})(\eta-\tau)\,|\hat{q}(\xi)|M\hat{q}(\tau-\xi)}{|\eta|^2}\,d\sigma(\xi)d\tau\,,$$

since if $\tau \in \Gamma_\varepsilon^-(\eta)$, then $|\phi(\tau)-\tau| < 4$, and by lemma 4.2, $|\hat{q}(\phi(\tau)-\xi)| \leq CM\hat{q}(\tau-\xi)$, for a certain constant $C > 0$. By lemma 4.6 and lemma 4.3, we have

$$\left\|\mathcal{F}^{-1}\left(\chi\, \widehat{J_1^2(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq C\left(\|q\|_{L^2}^2 + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{L^2}\right)\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\,.$$

This last estimate and (36) lead up to

$$\left\|\mathcal{F}^{-1}\left(\chi\, \widehat{J_1(q)}\right)\right\|_{\dot{W}^{\alpha,2}} \leq C\left[\|q\|_{L^2}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{L^2}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\right]\,.$$

In a similar way, by Calderón estimate, lemma 4.2 using that $|\hat{q}(\eta-\tau)| \leq CM\hat{q}(\eta-\phi(\tau))$, remark to lemma 4.5, lemma 4.6 and lemma 4.3 we may write

$$\|\mathcal{F}^{-1}\left(\chi\, \widehat{J_2(q)}\right)\|_{\dot{W}^{\alpha,2}} \leq C\left(\|q\|_{L^2}^2 + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{L^2}\right)\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\,,$$

and conclude that

$$\|\mathcal{F}^{-1}(\chi\,\widehat{Q''_\infty(q)})\|_{\dot{W}^{\alpha,2}} \leq C\left[\|\mathcal{F}^{-1}\left(\chi\, \widehat{J_1(q)}\right)\|_{\dot{W}^{\alpha,2}} + \|\mathcal{F}^{-1}\left(\chi\, \widehat{J_2(q)}\right)\|_{\dot{W}^{\alpha,2}} + \|J(q)\|_{\dot{W}^{\alpha,2}}\right]$$

$$\leq C\left[\|q\|_{L^2}^2\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{L^2}\|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\right]\,,$$

provided that $0 < \alpha + \varepsilon < 2$.

It is true that $Q''(q) = \mathcal{F}^{-1}\left([1-\chi]\,\widehat{Q''(q)}\right) + \mathcal{F}^{-1}\left(\chi\,\widehat{Q''(q)}\right)$, where the first term is a function belonging to the class $C^\infty(\mathbb{R}^2)$, and the second one belongs to $\dot{W}^{\alpha,2}(\mathbb{R}^2)$ taking $\varepsilon = s+1-\alpha$ in (23), (24) and (25). By lemma 4.2, $q \in \dot{W}^{-1+\varepsilon,2}(\mathbb{R}^2)$ (for $0 < \varepsilon < 1$) and replacing α by zero in (23), (24), (25) we get that $\mathcal{F}^{-1}\left(\chi\,\widehat{Q''(q)}\right) \in L^2(\mathbb{R}^2)$.

The remaining principal value term in (9) can be treated in a similar way. We do not include the proof to avoid tedious repetitions. To control those terms close to singularities we proceed similarly as we did for $Q''_\infty(q)$, compensating signs and using estimates for second differences. We have finished the proof of theorem 1.

$\square$

3. Proof of theorem 2

In order to avoid the control of the remainder term in the Born series, the following proposition gives, modulo a C^∞ function, the convergence of the Born series in $W^{\alpha,2}$, for $\alpha < s+\frac{1}{2}$.

Proposition 3.1. *Let $q \in W^{s,2}$ be a real-valued compactly supported function for $0 \leq s < 1$. Assume that $C_0 > 1$. Then, for any $\alpha \in \mathbb{R}$ such that $\alpha < s+\frac{1}{2}$:*

$$\|\widetilde{Q}_j(q)\|_{\dot{W}^{\alpha,2}} \leq C(s,\alpha)\,\|q\|_{L^2}\|q\|_{\dot{W}^{s,2}}^{-1}\,A(s,q,j)\,, \qquad (37)$$

where $\widetilde{Q}_j(q) = \mathcal{F}^{-1}\left(\chi^* \widehat{Q_j(q)}\right)$, $\chi^*(\xi) = 0$ *if* $|\xi| \leq C_0$, $\chi^*(\xi) = 1$ *if* $|\xi| > C_0$, $j \geq 4$ *and*

$$A(s, q, j) := \begin{cases} C_0^{\frac{5}{2}} \left[2^{\frac{7}{8}} C_0^{-\frac{3}{4}} \|q\|_{\dot{W}^{s,2}} \right]^j, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ C_0^{\frac{25}{8}} \left[2 C_0^{-\frac{7}{8}} \|q\|_{\dot{W}^{s,2}} \right]^j, & \text{if } \frac{1}{2} < s < 1. \end{cases}$$

Proof of proposition 3.1. We follow the lines of proposition 4.3 in [RV]. We lose some regularity in return for the gain of decay as a negative power of C_0. We write $R_\theta(k^2)(f)(x) := e^{-ik\theta \cdot x} R_+(k^2)(e^{ik\theta \cdot (\cdot)} f(\cdot))(x)$. By Cauchy-Schwartz inequality:

$$\|(qR_\theta(k^2))^{j-1}(q)\|_{L^1} \leq \|q\|_{L^2} \|R_\theta(k^2)(qR_\theta(k^2))^{j-2}(q)\|_{L^2}, \tag{38}$$

and applying successively the estimate for the resolvent given by lemma 3.4 in [R1] and the following inequality for Sobolev spaces due to Zolesio (see [G] and also Section 3.5 in [T]):

$$\|u\,v\|_{W^{s_3,p}(\mathbb{R}^n)} \leq \|u\|_{W^{s_1,p_1}(\mathbb{R}^n)} \|v\|_{W^{s_2,p_2}(\mathbb{R}^n)},$$

where $s_1, s_2, s_3 \geq 0$, $s_3 \leq s_1$, $s_3 \leq s_2$, $s_1 + s_2 - s_3 \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0$ and $p_j > p$, $j = 1, 2$, one can prove that

$$\|R_\theta(k^2)(qR_\theta(k^2))^{j-2}(q)\|_{L^2} \leq C k^{-1-\alpha_j} \|q\|_{\dot{W}^{s,2}}^{j-1}, \tag{39}$$

where

$$\alpha_j := \begin{cases} \frac{3}{4}(j-2) + \frac{s}{4}(j-1), & \text{if } s \leq \frac{1}{2}, \\ (j-3)\frac{3+s}{4} + 1, & \text{if } \frac{1}{2} \leq s < 1. \end{cases}$$

For all $h_j \in \mathbb{R}$ such that $h_j < \alpha_j$ it holds

$$\|\widetilde{Q}_j(q)\|_{\dot{W}^{h_j,2}}^2 \leq C 2^{2h_j} \int_{\frac{C_0}{2}}^{+\infty} k^{2h_j+1} \int_{S^1} \|(qR_\theta(k^2))^{j-1}(q)\|_{L^1(\mathbb{R}^2)}^2 d\sigma(\theta) dk \tag{40}$$

$$\leq C 2^{2h_j} \int_{\frac{C_0}{2}}^{+\infty} k^{2(h_j-\alpha_j)-1} dk \, \|q\|_{L^2}^2 \|q\|_{\dot{W}^{s,2}}^{2j-2}, \tag{41}$$

where we pass from (40) to (41) by formulae (38) and (39), and the last integral in k is convergent because of $h_j < \alpha_j$. So, we get

$$\|\widetilde{Q}_j(q)\|_{\dot{W}^{h_j,2}} \leq C \frac{2^{h_j}}{\sqrt{\alpha_j - h_j}} \left(\frac{C_0}{2}\right)^{h_j - \alpha_j} \|q\|_{L^2} \|q\|_{\dot{W}^{s,2}}^{j-1}. \tag{42}$$

Let $\varepsilon = \varepsilon(s, \alpha) := \left(s + \frac{1}{2}\right) - \alpha > 0$. We have

$$\|\widetilde{Q}_j(q)\|_{\dot{W}^{\alpha,2}} = \left[\int_{\{\xi:|\xi|>C_0\}} |\xi|^{2\alpha} \left|\widehat{Q_j(q)}(\xi)\right|^2 d\xi\right]^{\frac{1}{2}} \leq C_0^{s+\frac{1}{2}-\alpha_j} \|\widetilde{Q}_j(q)\|_{\dot{W}^{\alpha_j-\varepsilon,2}}$$

$$\leq C \frac{2^{\alpha_j - \varepsilon}}{\sqrt{\varepsilon}} \left(\frac{C_0}{2}\right)^{-\varepsilon} C_0^{s+\frac{1}{2}-\alpha_j} \|q\|_{L^2} \|q\|_{\dot{W}^{s,2}}^{j-1} = C(s,\alpha) \, 2^{\alpha_j} \, C_0^{\alpha-\alpha_j} \|q\|_{L^2} \|q\|_{\dot{W}^{s,2}}^{j-1},$$

where last inequality follows from formula (42) in the case $h_j = \alpha_j - \varepsilon$. Since

$$\alpha - \alpha_j < s + \frac{1}{2} - \alpha_j \leq \begin{cases} -\frac{3}{4}j + \frac{5}{2}, & \text{if } s \leq \frac{1}{2}, \\ -\frac{7}{8}j + \frac{25}{8}, & \text{if } \frac{1}{2} < s < 1, \end{cases}$$

and $2^{\alpha_j} \leq 2^{\frac{7}{8}j}$, if $s \leq \frac{1}{2}$ and $2^{\alpha_j} \leq 2^j$, if $\frac{1}{2} < s < 1$, we obtain (37).

$\square$

With the notation from proposition 3.1, the Born series (6) allows us to write

$$q_B - q = \mathcal{F}^{-1}\left(\chi^* \widehat{q_B - q}\right) + \mathcal{F}^{-1}\left((1-\chi^*)\widehat{q_B - q}\right) = \sum_{j=2}^{\infty} \widetilde{Q}_j(q) + \mathcal{F}^{-1}\left((1-\chi^*)\widehat{q_B - q}\right),$$

where $\mathcal{F}^{-1}\left((1-\chi^*)\widehat{q_B - q}\right)$ is C^∞. If we choose C_0 large enough, for example, taking

$$C_0 := \max\{10, 2^{\frac{7}{6}}\|q\|_{\dot{W}^{s,2}}^{\frac{4}{3}}, 2^{\frac{8}{7}}\|q\|_{\dot{W}^{s,2}}^{\frac{8}{7}}\} + 1,$$

it is true that $\sum_{j=4}^{+\infty} A(s,q,j) < +\infty$. From theorem 1 and [RV] we can write

$$\|\widetilde{Q}_2(q)\|_{\dot{W}^{\alpha,2}} \leq C\|q\|_{\dot{W}^{-\frac{1}{2},2}}\|q\|_{\dot{W}^{s,2}} \quad \text{and} \tag{43}$$

$$\|\widetilde{Q}_3(q)\|_{\dot{W}^{\alpha,2}} \leq C\left(\|q\|_{L^2}^2 + \|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}}\right)\|q\|_{\dot{W}^{s,2}}, \tag{44}$$

for all $\alpha < s + \frac{1}{2}$, and we have proved that

$$\sum_{j=4}^{+\infty} \|\widetilde{Q}_j(q)\|_{\dot{W}^{\alpha,2}} \leq C(s,\alpha)C_0^{-\frac{1}{2}} \frac{\|q\|_{L^2}\|q\|_{\dot{W}^{s,2}}^3}{1 - 2^{\frac{7}{8}}C_0^{-\frac{3}{4}}\|q\|_{\dot{W}^{s,2}}}, \tag{45}$$

if $0 \leq s \leq \frac{1}{2}$ (for $\frac{1}{2} < s < 1$, an analogous expression holds). We know that (43),(44),(45) remain true if $\alpha = 0$, obtaining the non-homogeneous Sobolev norm. We have finished the proof of theorem 2.

Acknowledgments

4. Appendix

Let it be the following submanifold of $\mathbb{R}^{2n}$ $V := \{(\eta,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi \cdot (\xi - \eta) = 0\}$. Then V can be considered from the point of view of the following spherical sections:

$$V = \{(\eta,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi \in \Gamma(\eta)\},$$

or the plane sections: $V = \{(\eta,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \eta \in \Lambda(\xi)\}$, where $\Gamma(\eta)$ and $\Lambda(\xi)$ are defined in (7) and (8). In this context, the following lemma from [RV] allows us to change the order of integration in ξ and η.

Lemma 4.1. *Let* $V \equiv \{(\eta,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi \cdot (\xi - \eta) = 0\}$. *Let* $d\sigma_\eta(\xi)$ *be the measure on* $\Gamma(\eta)$ *induced by the n-dimensional Lebesgue measure* $d\xi$ *and let* $d\sigma_\xi(\eta)$ *be the measure on* $\Lambda(\xi)$ *induced by the n-dimensional Lebesgue measure* $d\eta$. *Then*

$$d\sigma_\eta(\xi)d\eta = \frac{|\eta|}{|\xi|}\,d\sigma_\xi(\eta)d\xi.$$

The following lemma in [RV] is used several times in this work.

Lemma 4.2. *Assume that the support of q is contained in the unit ball. Then we have*

1) *If $\xi,\ \xi' \in \mathbb{R}^n$ satisfy $|\xi - \xi'| \leq 3$, then $|\hat{q}(\xi)| \leq CM\hat{q}(\xi')$.*

2) *$\|\hat{q}\|_{L^\infty} \leq C\|\hat{q}\|_{L^2}$.*

3) *For $0 < \beta < \frac{n}{2}$ and $s \in \mathbb{R}$, $\|q\|_{W^{s-\beta,2}} \leq C\|q\|_{W^{s,2}}$, where C depends on the size of the support of q.*

We want to indicate a

Definition. *Let $1 \leq p < +\infty$. We define the weights class A_p as the set of the non-negative locally integrable functions w that satisfy the so-called condition A_p, that is, that there exists a constant $C > 0$ independent of x and r so that*

$$\frac{1}{|B|^p} \int_B w(x)dx \left(\int_B w(x)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq C$$

for all ball B centred at $x \in \mathbb{R}^n$ and radius $r > 0$.

Indeed, next lemma 4.3, which is useful to bound the term $Q''_\infty(q)$, follows from estimates of the Hardy-Littlewood maximal operator and checking up on the function $|x|^{2s}$ belongs to the weighted class A_2 in two dimensions if $-1 < s < 1$.

Lemma 4.3. *Let q be a compactly supported function in $\dot{W}^{s,2}(\mathbb{R}^2)$. Hence there exists a positive constant C depending on the support of q such that for any $s \in \mathbb{R}$ with $|s| < 1$:*

$$\|\mathcal{F}^{-1}(M\hat{q})\|_{\dot{W}^{s,2}} \leq C\|q\|_{\dot{W}^{s,2}} \qquad and \qquad \|\mathcal{F}^{-1}(M\nabla\hat{q})\|_{\dot{W}^{s,2}} \leq C\|q\|_{\dot{W}^{s,2}}.$$

The following lemma is fundamental to control the term $Q'_I(q)$ by the formula (12).

Lemma 4.4. *Assume $\varepsilon > 0$ and $k \in \{1,2,...\}$. Let us denote*

$$F_k(\tau) := \int_{\Omega_k(\tau)} |\eta|^{2\alpha-3+2\varepsilon} \int \int_{I_k(\eta)} |\hat{q}(\eta - \tau' - \xi')|^2 d\sigma(\tau')d\sigma(\xi')d\sigma(\eta), \tag{4.1}$$

$$H(\tau) := \int_{\Lambda(\tau)} |\eta|^{2\alpha-3+2\varepsilon} \int_{\Gamma(\eta)} M\hat{q}(\eta - 2\tau')^2 d\sigma(\tau')\, d\sigma(\eta), \tag{4.2}$$

$$G(\tau) := \int_{\widetilde{\Lambda}(\tau)} |\eta|^{2\alpha-3+2\varepsilon} \int_{\Phi(\eta)} |M\hat{q}(\eta - \tau')|^2 d\sigma(\tau')d\sigma(\eta), \tag{4.3}$$

where

$$\Omega_k(\tau) := \left\{ \eta \in \Lambda(\tau) : |\eta - \tau| \geq C_1\, 2^{-k}|\eta| \right\}, \tag{4.4}$$

$$\Phi(\eta) := \{\ \tau' \in \Gamma(\eta) : |\tau'| \leq 1\} \quad and \quad \widetilde{\Lambda}(\tau) := \{\eta \in \Lambda(\tau) : |\eta| \geq 10\}, \tag{4.5}$$

and $I_k(\eta)$ is given in (16). Then

(i) *If $\tau \in \mathbb{R}^2 \setminus \{0\}$, $F_k(\tau) \leq C\,\|q\|^2_{\dot{W}^{\alpha-1+\varepsilon,2}}$, where C only depends on C_1.*

(ii) *For any $\tau \in \mathbb{R}^2$ such that $|\tau| > 1$, and for any $\alpha,\,\varepsilon$ so that $0 < \alpha + \varepsilon < 2$:*

$$H(\tau) \leq C\left(\|q\|^2_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|q\|^2_{L^2}|\tau|^{2\alpha-2+2\varepsilon}\right).$$

(iii) For any $\tau \in \mathbb{R}^2 \setminus \{0\}$ *such that* $|\tau| \leq 1$*, and provided that* $0 < \alpha + \varepsilon < 2$*:*

$$G(\tau) \leq C\|q\|^2_{\dot{W}^{\alpha-1+\varepsilon,2}} \,.$$

Remark. With respect to part (i) of this lemma, we need that $0 < C_1 < 2$, but in fact we always apply this lemma with $C_1 < 1$. Note that $F_k(\tau)$ is uniformly bounded in k.

Proof of (i)*.* For fixed τ we set an orthonormal reference $\{e_1, e_2\}$ of $\mathbb{R}^2$, for which $\tau = |\tau|e_1$. We write $\eta(s) = |\tau|e_1 + s\,e_2$, $s \in \mathbb{R}$. Let $h(s) := |\eta(s)| = (|\tau|^2 + s^2)^{\frac{1}{2}}$. Since $|s| = |\eta(s) - \tau| \geq C_1 2^{-k}h(s)$, we have $|s| \geq C_1 2^{-k}|\tau|$. It is true that $d\sigma(\eta(s)) = ds$. We have

$$F(\tau) = \int_{|s| \geq C_1 2^{-k}|\tau|} (h(s))^{2\alpha-3+2\varepsilon} \int \int_{I_k(\eta(s))} |\hat{q}(\eta(s) - \tau' - \xi')|^2 d\sigma(\tau')d\sigma(\xi')ds \,.$$

Take the change of variables given by

$$\xi' = \frac{\eta(s)}{2} + \frac{h(s)}{2}\,v \quad \text{and} \quad \tau' = \frac{\eta(s)}{2} + \frac{h(s)}{2}\,u \,,$$

with $u, v \in S^1$. It holds that $d\sigma(\xi') = Ch(s)d\sigma(v)$ and $d\sigma(\tau') = Ch(s)d\sigma(u)$. Since $|\eta(s) - \xi' - \tau'| \geq \frac{h(s)}{100}$, hence $1 + u \cdot v \geq \frac{1}{5000}$. It holds $|u - v| \leq 4 \cdot 2^{-k}$. We write

$$F(\tau) \leq C \int_{|s| \geq C_1 2^{-k}|\tau|} (h(s))^{2\alpha-1+2\varepsilon} \int_{S^1} \int_{A(v,k)} \left|\hat{q}\left(-\frac{h(s)}{2}(u+v)\right)\right|^2 d\sigma(u)d\sigma(v)ds$$

$$\leq C \sum_{j=1}^{2^k} \int_{|s| \geq C_1 2^{-k}|\tau|} (h(s))^{2\alpha-1+2\varepsilon} \int_{A_j} \int_{\widetilde{A}_j} \left|\hat{q}\left(-\frac{h(s)}{2}(u+v)\right)\right|^2 d\sigma(u)d\sigma(v)ds \,,$$

where $A(v,k) := \{u \in S^1 \,:\, |u - v| \leq 4 \cdot 2^{-k} \text{ and } 1 + u \cdot v \geq \frac{1}{5000}\}$ and $\{A_j \,:\, 1 \leq j \leq 2^k\}$ is a cover with finite overlapping of the circumference S^1. Each A_j has arc-length 2^{-k}. Also, $\widetilde{A}_j := \{u \in S^1 : |u - v| \leq 4 \cdot 2^{-k}, \ 1 + u \cdot v \geq \frac{1}{5000} \,, \text{ for some } v \in A_j\}$. Take the change of variables $u(\theta) = \cos\theta\,e_1 + \sin\theta\,e_2$, $\theta \in [0, 2\pi)$, $u \in S^1$, with $d\sigma(u(\theta)) = d\theta$. By Fubini's theorem,

$$F(\tau) \leq C \sum_{j=1}^{2^k} \int_{A_j} \int_{|s| \geq C_1 2^{-k}|\tau|} \int_{\{\theta \,:\, u(\theta) \in \widetilde{A}_j\}} (h(s))^{2\alpha-1+2\varepsilon} \left|\hat{q}\left(-\frac{h(s)}{2}(u+v)\right)\right|^2 d\theta ds d\sigma(v) \,.$$

For any frozen $j \in \mathbb{Z}$ with $1 \leq j \leq 2^k$ and $v \in A_j$, take the change of variables $(s, \theta) \rightarrow \lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ given by

$$\lambda = -\frac{h(s)}{2}(u(\theta) + v) = -\frac{h(s)}{2}((\cos\theta + v_1)e_1 + (\sin\theta + v_2)e_2) \,.$$

It holds $dsd\theta = \frac{4}{|s(1 + u(\theta) \cdot v)|}\,d\lambda$. For any $j \geq 1$, we consider the proper cone

$$H_j := \left\{r\left(\frac{u+v}{2}\right) : r < 0, \, v \in A_j, \, u \in \widetilde{A}_j\right\} \,.$$

Since $1 + u(\theta) \cdot v \geq C$, $h(s) \sim |\lambda|$. We know that for $|s| \geq C_1 2^{-k}|\tau|$ we also have that $|s| \geq C(C_1)2^{-k}h(s)$, $\sigma(A_j) \sim 2^{-k}$ and the family $\{H_j : 1 \leq j \leq 2^k\}$ has finite overlap with constant independent of k. Then

$$F(\tau) \leq C\,2^k \sum_{j=1}^{2^k} \int_{A_j} \int_{H_j} |\lambda|^{2\alpha-2+2\varepsilon} |\hat{q}(\lambda)|^2 d\lambda\, d\sigma(v) = C \sum_{j=1}^{2^k} \int_{H_j} |\lambda|^{2\alpha-2+2\varepsilon} |\hat{q}(\lambda)|^2 d\lambda$$

$$\leq C \int_{\mathbb{R}^2} |\lambda|^{2\alpha-2+2\varepsilon} |\hat{q}(\lambda)|^2 d\lambda = C\,\|q\|^2_{\dot{W}^{\alpha-1+\varepsilon,2}} \,.$$

$\square$

Proof of (ii). We follow the same lines of the previous point but now we do not need the finite overlapping cover for S^1. The variable s takes real values in all the line. In the same way, take the change $\tau' = \frac{\eta(s)}{2} + \frac{|\eta(s)|}{2}u$, with $u \in S^1$ and we parametrize u by $\theta \in [0, 2\pi)$. We take the change of variables $(s, \theta) \to \lambda = (\lambda_1, \lambda_2)$ given by

$$\lambda = \eta(s) - 2\tau'(\theta) = -|\eta(s)|u(\theta) = -|\eta(s)|(\cos\theta\, e_1 + \sin\theta\, e_2)\,.$$

Now $dsd\theta = \frac{d\lambda}{|s|} = \frac{d\lambda}{(\,|\eta(s)|^2 - |\tau|^2\,)^{\frac{1}{2}}}$, and $|\lambda| = |\eta(s)| \geq |\tau|$. So, we obtain that

$$H(\tau) \leq \left(\int_{\{\lambda : |\lambda| \geq (1+|\tau|^2)^{\frac{1}{2}}\}} + \int_{\{\lambda : |\tau| \leq |\lambda| \leq (1+|\tau|^2)^{\frac{1}{2}}\}} \right) |\lambda|^{2\alpha - 2 + 2\varepsilon} M\hat{q}(\lambda)^2 \frac{d\lambda}{(|\lambda|^2 - |\tau|^2)^{\frac{1}{2}}}$$
$$=: J_1 + J_2\,.$$

The first integral has no difficulties, indeed, $J_1 \leq C\|q\|^2_{\dot{W}^{\alpha-1+\varepsilon,2}}$, by lemma 4.3 provided that $0 < \alpha + \varepsilon < 2$. By lemma 4.2, $M\hat{q}(\lambda) \leq C\|q\|_{L^2}$ and taking polar coordinates we get $J_2 \leq C\|q\|^2_{L^2}|\tau|^{2\alpha - 2 + 2\varepsilon}$.

$\square$

Proof of (iii). Let τ be in $\mathbb{R}^2 \setminus \{0\}$ such that $|\tau| \leq 1$. Following the same scheme as the last point, we parametrize the variable η by $s \in \mathbb{R}$, take the change $\tau' = \frac{\eta(s)}{2} + \frac{|\eta(s)|}{2}u$, with $u \in S^1$, and parametrize u by $\theta \in (0, 2\pi]$. Finally take the change $(s, \theta) \to \lambda = (\lambda_1, \lambda_2)$, given by $\lambda = \eta(s) - \tau'(\theta) = \frac{1}{2}\left[(|\tau| - |\eta(s)|\cos\theta)e_1 + (s - |\eta(s)|\sin\theta)e_2\right]$. In this case, the Jacobian for this change is $dsd\theta = \frac{2}{|\lambda_2|}d\lambda_1 d\lambda_2$. The condition $|\tau|, |\tau'(\theta)| \leq 1$ guarantees that the angle between $\tau - \eta(s)$ and $\tau'(\theta) - \eta(s)$ is uniformly bounded by an acute angle. Remember that $|\eta(s)| \geq 10$. So, $|\lambda_2| \sim |\lambda|$. That condition also implies that a positive constant $C < 1$ exists such that $|\lambda| = |\eta(s) - \tau'(\theta)| \geq C|\eta|$, hence $|\eta(s)| \sim |\lambda|$. It holds

$$G(\tau) \leq C \int_{\{\lambda \in \mathbb{R}^2 : |\lambda| > C'\}} |\lambda|^{2\alpha - 2 + 2\varepsilon} M\hat{q}(\lambda)^2 \frac{d\lambda}{|\lambda|} \leq C \left\|\mathcal{F}^{-1}(M\hat{q})\right\|^2_{\dot{W}^{\alpha-1+\varepsilon,2}}\,,$$

and lemma 4.3 ends the proof.

$\square$

The following lemma becomes essential to bound the terms $Q''_j(q)$ in (22).

Lemma 4.5. *Let*

$$\widehat{Q''_\delta(q)}(\eta) := \chi_{(\delta^{-1}, +\infty)}(|\eta|)\frac{1}{|\eta|^3} \int_{\Gamma_\delta(\eta)} \int_{\Gamma(\eta)} |\hat{q}(\xi)\hat{q}(\eta - \tau)\hat{q}(\tau - \xi)|\, d\sigma(\xi)d\tau\,, \tag{4.6}$$

where $\Gamma_\delta(\eta)$ *is the annulus given by*

$$\Gamma_\delta(\eta) := \left\{ \tau \in \mathbb{R}^2 : \left| |\tau - \frac{\eta}{2}| - \frac{|\eta|}{2} \right| \leq \delta|\eta| \right\}\,. \tag{4.7}$$

Then there exist δ_0, $C(\delta_0)$, β *so that* $\delta_0 > 0$, $C(\delta_0) > 0$, $\beta > 1$ *and for any* δ *with* $0 < \delta \leq \delta_0$:

$$\left\|Q''_\delta(q)\right\|_{\dot{W}^{\alpha,2}} \leq C(\delta_0)\delta^\beta \|q\|_{\dot{W}^{\alpha-1+\varepsilon,2}} \left(\|q\|_{L^2}\|q\|_{\dot{W}^{-\frac{1}{2},2}} + \|q\|^2_{L^2} \right)\,,$$

where $\alpha \in \mathbb{R}$ *and* $\varepsilon > 0$ *satisfy that* $0 < \alpha + \varepsilon < 2$.

We omit the proof of this lemma. We know that $d\sigma_\eta(\tau) = \lim\limits_{\delta \to 0} \dfrac{1}{\delta|\eta|} \chi_{\Gamma_\delta(\eta)}(\tau) d\tau$, where $d\sigma_\eta(\tau)$ denotes the measure on $\Gamma(\eta)$ induced by $d\tau$. According to this, $\widehat{Q''_\delta(q)}(\eta) \sim \delta \widehat{Q'(q)}(\eta)$, if $0 < \delta < \delta_0 \ll 1$. Heuristically the estimate for $Q''_\delta(q)$ is the one for $Q'(q)$ multiplied by δ. We have to pay with a fraction of derivatives in $\|Q'(q)\|_{\dot{W}^{\alpha,2}}$ in order to gain the factor δ^β with $\beta > 1$. So, the reader must not be surprised by the lemma whose proof follows the lines of the estimate of $Q'(q)$.

Remark 4.1. If we substitute (4.6) for

$$\widehat{Q''_\delta}(f,g,h)(\eta) := \chi_{(\delta^{-1},+\infty)}(|\eta|)\frac{1}{|\eta|^3} \int_{\Gamma_\delta(\eta)} \int_{\Gamma(\eta)} \left| \hat{f}(\xi)\hat{g}(\eta - \tau)\hat{h}(\tau - \xi) \right| d\sigma(\xi) d\tau,$$

with $f,g,h \in W^{s,2}$ and $-1 < s < 1$, just imitating the proof of the control of the spherical term $Q'(q)$, we get that there exist $\beta > 1$ and $C(\delta_0) > 0$ so that

$$\|Q''_\delta(f,g,h)\|_{\dot{W}^{\alpha,2}} \le C(\delta_0)\delta^\beta \|g\|_{L^2}\big[\|f\|_{L^2}\|h\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|f\|_{\dot{W}^{-\frac{1}{2},2}}\|h\|_{\dot{W}^{\alpha-1+\varepsilon,2}}$$
$$+ \|h\|_{L^2}\|f\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\big],$$

for any δ such that $0 < \delta \le \delta_0$.

Lemma 4.6. *Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that $0 < \alpha + \varepsilon < 2$. Let $f,g,h \in W^{s,2}$ for all $s \in \mathbb{R}$ with $-1 < s < 1$. Let*

$$\widehat{Q''_*(q)}(\eta) := \chi(\eta) \frac{1}{|\eta|^2} \int_{\Gamma_\infty(\eta)} \int_{\Gamma(\eta)} \left| \hat{f}(\xi)\hat{g}(\eta - \tau)\hat{h}(\tau - \xi) \right| d\sigma(\xi) d\tau,$$

where $\Gamma_\infty(\eta)$ is the annulus given by (21). Then there exists a constant $C > 0$ such that

$$\|Q''_*(q)\|_{\dot{W}^{\alpha,2}} \le C\big[\|f\|_{L^2}\|g\|_{L^2}\|h\|_{\dot{W}^{\alpha-1+\varepsilon,2}} + \|f\|_{\dot{W}^{-\frac{1}{2},2}}\|g\|_{L^2}\|h\|_{\dot{W}^{\alpha-1+\varepsilon,2}}$$
$$+ \|f\|_{\dot{W}^{\alpha-1+\varepsilon,2}}\|g\|_{L^2}\|h\|_{L^2}\big].$$

Remark 4.2. Compare this lemma with remark to lemma 4.5 when, morally, $\delta \sim |\eta|^{-1}$. Consider that we work with a similar term with an annulus with $\delta \sim |\eta|^{-1}$, but in the estimate we claim the same gain of derivatives as in remark 4.1.

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